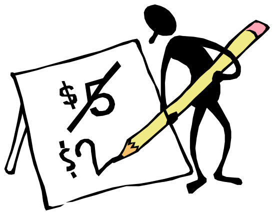
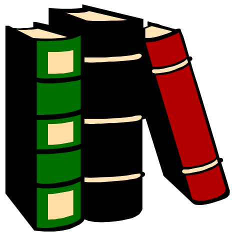
Matrix Algebra and Applications





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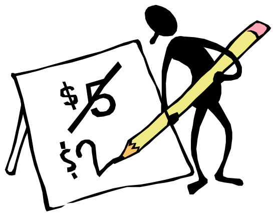

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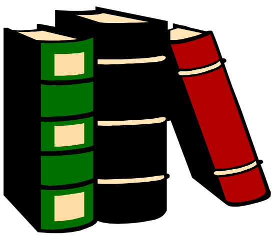

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1 Algebraic Foundations

1.1 Vectors and Vector spaces

What is a vector ? on meets vectors within some physical context as force, velocity, position and so on;
they have both magnitude and direction;
two- or three-dimensional space;
generalization to n-space is necessary

Let be R the set of all real numbers. Than we consider the Cartesian Product $R^n = \{(x_1,\ldots,x_n) | x_i \in R\}$.
Any real n-tupel

$$\vec{x} = (x_1,\ldots,x_n) \in R^n$$

is called a n-dimensional vector (n is called the dimension).

Remark: if there is no confusion we leave out the arrow

particulary: $n = 2$: $a = (2,5),\ b = (3,-7)$ (two-dimensional)
$n = 3$: $x = (1,1,1),\ y = (3,-1,2)$ (three-dimensional)

Graphical representation in 3-dimensions: arrows

Cartesian Coordinates System

we refer to a system of three straight line axis x, y, z in space which are at right angles to each other; the crossing point is the origin 0

we denote by $e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)$

vectors from the origin directed along the x, y, z axis respectively.
So any vector from the origin may be written in the form

$$v = (x,y,z) = xe_1 + ye_2 + ze_3$$

in the Cartesian Coordinates System $\{0;e_1,e_2,e_3\}$.

vector from a point $A(a_x, a_y, a_z)$ to another point $B(b_x, b_y, b_z)$

$$v = (b_x - a_x, b_y - a_y, b_z - a_z)$$

Magnitude and Direction of a vector $v = (x, y, z)$

$|v| = \sqrt{x^2 + y^2 + z^2}$ is called the **magnitude** of v (length, norm)

direction of v is represented by the angles α, β, γ between v and the three axis respectively; it is

$$cos(\alpha) = \frac{x}{|v|},\ cos(\beta) = \frac{y}{|v|},\ cos(\gamma) = \frac{z}{|v|}$$

notice: the location of a vector is not specified, only its magnitude and direction

Operations with vectors in real n-space

Addition $x + y = (x_1 + y_1,\ldots,x_n + y_n)$ with $x, y \in R^n$

Scalar Multiplication $\alpha \cdot x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$, $\alpha \neq 0, \alpha \in R$ and $x \in R^n$

(in R^3 the length of vector αx is $|\alpha|$ times the length of x; direction of αx is the same as that of x if $\alpha > 0$, and the opposite if $\alpha < 0$)

Difference $x - y = x + (-1) \cdot y$

Norm of a vector $\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$

notice: The norm of a vector is a function $\vartheta : R^n \to R$ with the properties

(i) $\vartheta(tx) = |t| \cdot \vartheta(x)$

(ii) $\vartheta(x + y) \leq \vartheta(x) + \vartheta(y)$

(iii) $\vartheta(x) \geq 0$ and $\vartheta(x) = 0$ for $x = 0$

Dot product $x \bullet y = x_1 y_1 + \ldots + x_n y_n$ (scalar product, inner product)

notice: The inner product is a function $\varphi : R^n \times R^n \to R$ with the properties

(i) $\varphi(x, y) = \varphi(y, x)$

(ii) $t\varphi(x, y) = \varphi(tx, y) = \varphi(x, ty)$

(iii) $\varphi(x + y, z) = \varphi(x, z) + \varphi(y, z)$

(iv) $\varphi(x, x) > 0,\ x \neq 0$

Angle between two vectors $x \bullet y = \|x\| \cdot \|y\| \cdot \cos\alpha$

notice: This is possible because of the relation

$$-1 \le \frac{x \bullet y}{\|x\| \cdot \|y\|} \le +1$$

two vectors x, y are called orthogonal iff their dot product is zero:

$$x \bullet y = 0$$

Cross product (only in 3-space) vector $v = x \times y$ with

(i) length $|v| = |x| \cdot |y| \cdot \sin(x,y)$
(ii) v is orthogonal to both x and y
(iii) x, y, v form a right-handed system

we have (without derivation)

$$x \times y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} e_1 - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} e_2 + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} e_3$$

$|x \times y|$ = the area of a parallelogram with sides x and y

two vectors x, y are parallel iff their cross product is zero

properties $x \times y = -y \times x$
$\alpha(x \times y) = (\alpha x) \times y = x \times (\alpha y)$, α is a scalar
$x \times (y + z) = x \times y + x \times z$

Special vectors $0 := (0,\ldots,0)$ zero vector (direction is not defined)
$e_i := (0,\ldots,0,1,0,\ldots,0)$, $1 \le i \le n$

Vector space $(V,+,*,K)$

V **a nonempty set with elements** $a,b,c,...$ **(vectors)**
$K=(K,\oplus,\otimes)$ **a field with elements** $\alpha,\beta,...$ **(scalars)**

Binary operation $+: V\times V \to V$
„Multiplication" $*: K\times V \to V$

(A) The structur $(V,+)$ **forms a commutative group:**

(i) $x+y=y+x$ (commutativity)
$x+(y+z)=(x+y)+z$ (associativity)
(ii) $x+0=x, \forall x\in V$ (neutral element 0)
(iii) $x+(-x)=0$ (inverse element)

(B) The „Multiplication" fullfills the properties:

(i) $\alpha * x \in V$ for $x\in V, \alpha\in K$
(ii) $\alpha*(\beta*x)=(\alpha\otimes\beta)*x$
(iii) $(\alpha\oplus\beta)*x=\alpha*x+\beta*x$ and $\alpha*(x+y)=\alpha*x+\alpha*y$
(iv) $1*x=x$ for all $x\in V$

A Vector space with a inner product we call an **Eucledian Vectorspace.**

Examples

(1) Real Numbers form a vector space $(R,+,\cdot,R)$
(2) The n-dimensional real vectors $x,y,z,...$ with the real scalars $\alpha,\beta,\gamma,...$ define an eucledian vectorspace $(R^n,+,\cdot,R)$.
(3) The real (m,n)-matrices form a vector space $(A_{m,n},+,*,R)$.

Linear independence of vectors

The set of vectors $a_1,...,a_m \in V$ are called linearly independent iff

$$\alpha_1 * a_1 + ... + \alpha_m * a_m = 0 \text{ implies that } \alpha_1 = 0,...,\alpha_m = 0 .$$

The set of all vectors of the form

$$v := \alpha_1 * a_1 + ... + \alpha_m * a_m \text{ for some scalars } \alpha_1,...,\alpha_m \in K$$

built the vector space $S(a_1,...,a_m)$; it is a sub vector space of $(V,+,*,K)$.

Basis of $(V,+,*,K)$

The set $B = \{b_1,...,b_n\}$ of vectors of $(V,+,*,K)$ is called a basis of the vector space if

(a) $S(b_1,...,b_n) = V$,
(b) $\{b_1,...,b_n\}$ are linearly independent .

Examples

(1) The set $\{e_1,...,e_n\}$ is a basis of $(R^n,+,\cdot,R)$.
(2) The set $\{E_{11},...,E_{1n},E_{21},...,E_{2n},...,E_{m1},...,E_{mn}\}$ of (m,n)-matrices E_{ij} with $e_{ij} = 1$ and $e_{ij} = 0$ otherwise form a basis of the vector space $(A_{m,n},+,*,R)$.

1.2 Matrices and Determinants

What is a matrix ? on needs matrices to simplify a large number of quantities;
in a certan manner its a collection of several n-dimensional vectors

A matrix A is a rectangular array of quantities in m rows and n column like follows

$$A = \begin{pmatrix} a_{11} & a_{12} & .... & .... & a_{1n} \\ a_{21} & a_{22} & .... & .... & a_{2n} \\ . & . & & & . \\ . & . & & & . \\ a_{m1} & a_{m2} & .... & .... & a_{mn} \end{pmatrix} , \qquad A_{m\times n} = \{a_{ij}\}$$

$m \times n$ is called the form or dimension of the matrix

$M^K_{m\times n}$ the set of all $(m\times n)$-matrices with elements from the field K

Two matrices A,B are said to be equal if they are of the same dimension and if their corresponding elements are equal.

The quantities a_{ij} are called the elements and can be arbitrary objects like real numbers, complex numbers, functions, differential operators or even matrices themselves.

$m = n$: square matrix of order n $\rightarrow$ $M^K_{n\times n}$

$A_{m\times n} = \{a_{ij}\}$ is a diagonal matrix of order n if the only nonzero elements lie on the main diagonal : $A = diag(a_{11},...,a_{nn})$ (that is $a_{ij} = 0,\ i \neq j$).
in particular: $I_n = diag(1,...,1)$ is called Identity (unit) matrix

Operations with matrices from $M_{m\times n}^{K}$

Matrix Addition $A+B=\{a_{ij}+b_{ij}\}$ with $A,B\in M_{m\times n}^{K}$ (the same dimension)

Scalar Multiplication $\alpha A=\{\alpha\cdot a_{ij}\}$, $\alpha\in K$, $A\in M_{m\times n}^{K}$, $(\alpha A)\in M_{m\times n}^{K}$

especially: $-A=(-1)A$ additive inverse of A

Difference $A-B=A+(-B)$

Matrix Multiplication $C=A\cdot B$ with $C=\{c_{ij}\}\in M_{m\times p}^{K}$ and

$$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj} \; ,$$

suppose $A\in M_{m\times n}^{K}$ and $B\in M_{n\times p}^{K}$ (conformable matrices)

conformable for multiplication that is the number of columns of A is equal to the number of rows of B

notice: matrix multiplication means to carry out the scalar product of each row of the first matrix with each column of the second one

Some properties $A+B=B+A$, $A+(B+C)=(A+B)+C$

$A+0=A$, $A+(-A)=0$

$\alpha(\beta A)=(\alpha\beta)A$, $(\alpha+\beta)A=\alpha A+\alpha B$

$1A=A$, $0A=0$

$A\cdot B\neq B\cdot A$, $A\cdot(B+C)=A\cdot B+A\cdot C$

$A\cdot(B\cdot C)=(A\cdot B)\cdot C$, $I\cdot A=A\cdot I=A$

Elementary row or column operations

(i) interchanging two rows / columns
(ii) multiplying the elements of one row / column by a non-zero real number
(iii) adding to the elements of one row / column, any multiple of the corresponding elements of another row / column

Elementary Matrix Matrix which can be obtained from the identity matrix by performing just one elementary operation

Two matrices $A, B \in M^K_{m \times n}$ are equivalent (symbol: $A \equiv B$) if B can be obtained from A by a finite number of elementary operations op_i, $1 \leq i \leq k$:

$$A \equiv B \quad :\Leftrightarrow \quad B = op_k(op_{k-1}(...op_1(A)...))$$

helpfull propery for matrix calculations:

Every regular square matrix with elements from a field $K = (K, +, *)$ can be represent by a finite product of elementary matrices.

Some special Matrices

A^T transposed matrix of A, $a^T_{ij} = a_{ji}$

$A = A^T$ symmetric matrix

$\overline{A}^T$ adjoint matrix of A

$\overline{A}^T = A$ hermitian matrix

Determinant a scalar quantity associated with a square matrix $A_{n\times n}$

function $det\ A{:}\ A_{n\times n} \rightarrow s\ \ with\ \ s \in R,\ a_{ij} \in R$ (like a rule)

$n = 1$: $det\ A = a_{11}$
$n = 2$: $det\ A = a_{11}a_{22} - a_{21}a_{12}$
$n \geq 3$: more generally

Let be $s = (k_1 k_2 ... k_n)$ a permutation of the set $N_n^+ := \{1,2,...,n\}$, S_n the set of all permutations of N_n^+ and $I(s)$ the number of inversions of s. Then the determinant of A is

$$det\ A := \sum_{s \in S_n} (-1)^{I(s)} a_{1k_1} a_{2k_2} ... a_{nk_n}$$

in particular: $n = 3 \ \Rightarrow$ Rule of Sarrus

$$det\ A = det\ A^T$$

some notations:

minor M_{ij} of the element a_{ij} in A is the determinant of the order n-1 that survives when the *i*th row and the *j*th colummn are struck out

cofactor A_{ij} of the element a_{ij} in A is defined as

$$A_{ij} = (-1)^{i+j} M_{ij}$$

cofactor expansion of a determinant by its *i*th row

$$det\ A = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in} \qquad \text{(Theorem of Laplace)}$$

notice: if all elements in a row are zero elements then det A = 0

det A = 0 then A is called singular otherwise regular

Example to compute a determinant:

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{pmatrix} \qquad \text{det A = ?}$$

using the expansion with $i = 1$ we have

$$det\ A \ = \ a_{11} A_{11} + a_{12} A_{12} + a_{i13} A_{13}$$

$$= \ 0 \cdot (-1)^2\, M_{11} + 2 \cdot (-1)^3\, M_{12} + (-1) \cdot (-1)^4\, M_{13}$$

$$M_{11} = \begin{vmatrix} 3 & 5 \\ 0 & -4 \end{vmatrix} \ ,\ M_{12} = \begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix} \ ,\ M_{13} = \begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix}$$

$$det\ A \ = \ 0(-12 + 0) - (2)(-16 - 10) + (-1)(0 - 6) = 58$$

Properties of determinants

(i) Two equal rows or columns

$$det(a_1, \dots, a_r, \dots, a_r, \dots, a_n) = 0$$

(ii) Multiple of a row or column by a scalar

$$det(a_1, \dots, ta_r, \dots, a_n) = t \cdot det(a_1, \dots, a_r, \dots, a_n)$$

(iii) Interchange of two rows or columns

$$det(a_1, \dots, a_r, \dots, a_s \dots, a_n) = -det(a_1, \dots, a_s, \dots, a_r \dots, a_n)$$

(iv) Factor in a row or column

$$det(a_1, \dots, a_r, \dots, ta_r, \dots, a_n) = 0$$

(v) Adding multiples of a row or column

$$det(a_1, \dots, a_r + ta_s, \dots, a_n) = det(a_1, \dots, a_r, \dots, a_n)$$

Inverse Matrix Let A be of order n. Any Matrix Y of order n which helds the equation

$$A \cdot Y = Y \cdot A = I$$

is called the inverse of A, denoted by $Y \equiv A^{-1}$.
The matrix A is then said to be invertible.

notice: If A is invertible, then A^{-1} is unique.

A has an inverse iff $det\, A \neq 0$

Some poperties

$$(A^{-1})^{-1} = A$$
$$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$$
$$(A^T)^{-1} = (A^{-1})^T$$

How to obtain the inverse of a matrix A ?

(1) compute all cofactors A_{ij} of the element a_{ij} in A

$$A_{ij} = (-1)^{i+j} M_{ij} \;,\; \text{i,j=1,...,n}$$

(2) compute the determinant of A by cofactor expansion

$$det\, A = a_{i1} A_{i1} + a_{i2} A_{i2} + ... + a_{in} A_{in} \quad \text{, i fix}$$

(3) compute $$A^{-1} = \frac{1}{det\, A} \begin{pmatrix} A_{11} & \dots & \dots & A_{n1} \\ A_{12} & \dots & \dots & A_{n2} \\ . & . & . & . \\ A_{1n} & \dots & \dots & A_{nn} \end{pmatrix}$$

But in practice we use **numerical methods !!!**

Rank maximal number of linear independent rows or columns of the matrix $A_{m\times n}$

Obviously we have: $rk(A) \le min(m,n)$

Criterion of Frobenius

$rk(A) =$ the order r of the largest square submatrix with non-zero determinant

Rank criterium Let be obtained the matrix B from a matrix A by a finite sequence of elementary row or column operations. Then the matrices A,B have the same rank:

$$A,B \in M^K_{m\times n} \text{ with } A \equiv B \;\Rightarrow\; rk(B) = rk(A)\ .$$

(the maximal number of linear independent rows or columns has not changed)

Computational method

1. Transform the matrix A by elementary row or column operations so that zero elements arise (in particular we get an upper triangular matrix)

2. Compute the order r of the largest square submatrix with non-zero determinant (use the properties of determinants)

More generally: Every matrix $A \in M^K_{m\times n}$ with rank $rk(A) = r$ is equivalent to the matrix of the form

$$I_{m\times n} := \begin{pmatrix} I_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix}$$

$\Rightarrow$ Numerical Algorithm: apply elementary operations on $A \in M^K_{m\times n}$ as far the matrix $I_{m\times n}$ is reached

1.3 Algebraic Equations

What is an algebraic equation ?

Equation $f(x)=0$ is said to be **algebraic** (polynomial) over the field $(K,+,*)$, if the function f is expressible in the form

$$f(x)=a_n x^n + a_{n-1}x^{n-1}+...+a_1 x + a_0 \ , \ a_n \neq 0, \ a_i \in K$$

$n \in N$ is called the degree (dimension);

otherwise f is said to be **transcendental**.

example: $3x^2 + x - 2 = 0$, $-2x^4 + 5x^2 + x = 0$
$e^x - sin(x) = 0$, $x + 1 - cos(x) = 0$

particulary: $n=1$: $f(x) = mx + n$ (linear, first-degree polynomial)
$n=2$: $f(x) = ax^2 + bx + c$ (nonlinear, second-degree)

Graphical representation: straight line, parabolas

Special forms

zeros equation	$f(x)=0$	solution leads to zeros
fixpoint equation	$f(x)=x$	solution leads to fixpoints that mean points of intersection between the graphs of f(x) and g(x)=x

The collection of all the solutions is called the solution set.

Systems of equations more then one equation and more then one unknown, say m equations in n unknowns

$$
\begin{aligned}
&f_1(x_1,x_2,\ldots,x_n)=0\\
&f_2(x_1,x_2,\ldots,x_n)=0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&f_m(x_1,x_2,\ldots,x_n)=0
\end{aligned}
$$

if there exist solutions: the system is consistent (otherwise inconsistant)

if there is precisely one solution: unique solution (otherwise nonunique)

important special case: **Linear algebraic Systems**

$$
\begin{aligned}
&a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1\\
&a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n=b_2\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n=b_m
\end{aligned}
\qquad \text{with} \quad a_{ij},b_i\in K,\ 1\le i\le m,\ 1\le j\le n
$$

or in matrix notation: $A_{m\times n}\cdot x=b$ with $A\in M^K_{m\times n},\ b\in K^m,\ x\in K^n$

Solution methods:

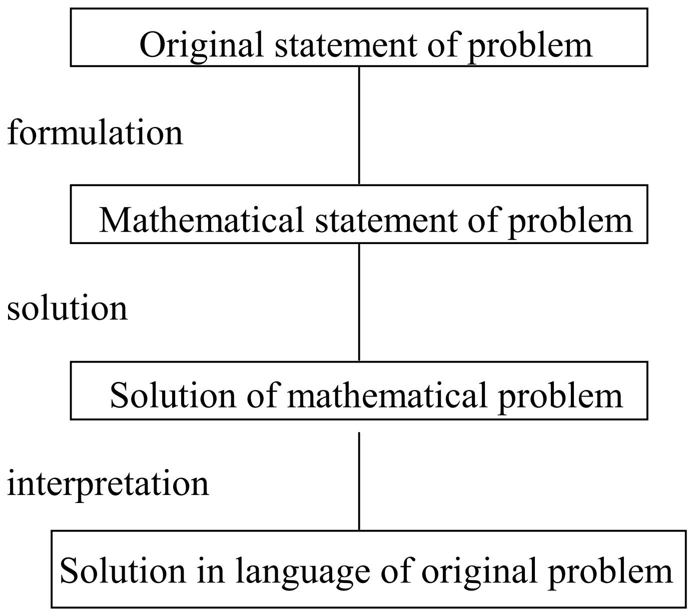
1. elemination methods: Gauss Elimination (most important)
2. iterative methods: Jacobi method, Gauss-Seidel iteration

notice: Two linear systems are equivalent if one can be obtained from the other by a finite number of elementary operations.

elementary operations: interchange of two equations, addition of a multiple of one equation to another, multiplication of an equation by a nonzero constant

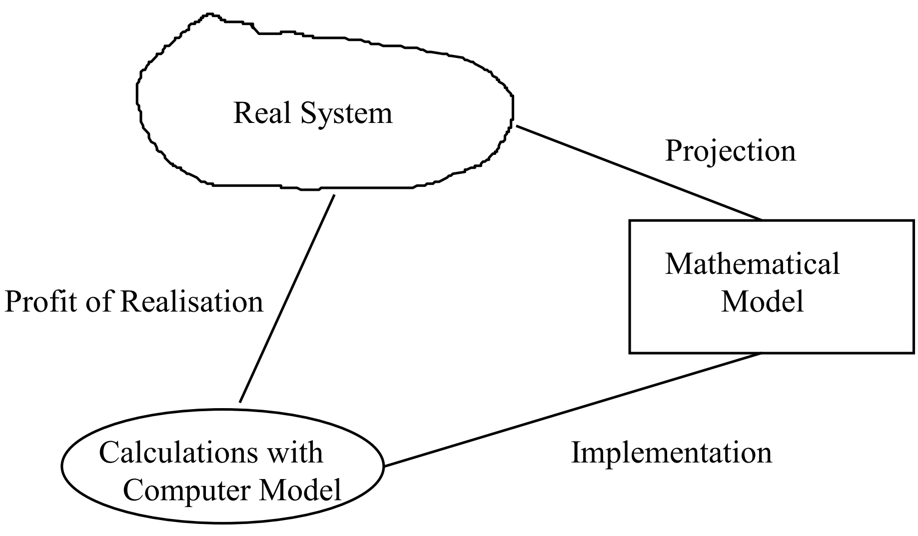
Solving applied problems

Diagram with the major conceptuel steps to solve applied problems



Simulation

Projection of a real dynamic system in a mathematical model
Implementation and Calculations with the Computer model
Interpretation of the results
Profit of Realisation (perception) with the behavior of the real system



Solution of a nonlinear System

We consider the vector equation $f(x) = 0$.

In the process of solving a problem or setting up a mathematical model, we encounter an equation of this form. *(Real Poblem → Mathematical Model)*

How to get a solution of this zeros equations ?

Problem: unfortunately it is offen difficult or impossible to calculate this zeros exactly

Numerical Methods *(Mathematical Model → Computer Model)*

Solution idea: iteration process

repeating a procedure over and over, starting with an approximation value, until a desired degree of accuracy of the approximation is obtained

$x_{k+1} = \Phi(x_k),\ k = 0,1,\ldots$, with starting value $x_0 \in I(x^*)$

$\Rightarrow$ sequence $\{x_k\}$ with $\lim\limits_{k\to\infty} x_k = x^*$ and $f(x^*) = 0$

Newton´s method

Approximation scheme for Newton´s Method

$$x_{k+1} = x_k - [f'(x_k)]^{-1} f(x_k) \ , \ k = 0,1,\ldots$$

and initial value $x_0 \in U(x^*)$

particulary: $m = n = 1$: $\quad x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)} \ , \ k = 0,1,\ldots$

(if $f'(x_k) \neq 0$)

Example: **Calculate $\sqrt{3}$ using the Newton´s method**

Take $f(x) = x^2 - 3$ so the solution of $f(x) = 0$ is the value $\sqrt{3}$

$f'(x) = 2x$

$\Rightarrow$ Approximation scheme of Newton´s method is

$$x_{n+1} = x_n - \frac{x^2{}_n - 3}{2x_n} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right) \;,\; n = 0,1,\ldots$$

initial guess: $f(1) = -2 \;, f(2) = 5 \;\Rightarrow\; x^* \in [-2,5] =: I(x^*)$
choose $x_0 = 1 \in I(x^*)$

iteration process:

$$x_1 = \frac{1}{2}\left(1 + \frac{3}{1}\right) = 2 \;,\; x_2 = \frac{1}{2}\left(2 + \frac{3}{2}\right) = 1.75$$

$$x_3 = \;?\;,\; x_4 = \;?\;,\; \ldots.. \quad \Rightarrow \{x_k\}$$

stop if $\left|x_{k+1} - x_k\right| < 0.001$

Now we are coming back to the general case of m equations in n unknowns of the form

$$\begin{aligned} &f_1(x_1,x_2,\dots,x_n)=0 \\ &f_2(x_1,x_2,\dots,x_n)=0 \\ &\dots\dots\dots\dots\dots\dots \\ &f_m(x_1,x_2,\dots,x_n)=0 \end{aligned} \quad \Leftrightarrow \quad f(x)=0 \ \text{ with } \ x\in K^n,\ f:K^n\to K^m$$

Newton´s method

Approximation scheme for Newton´s Method

$$x_{k+1}=x_k-[f'(x_k)]^{-1}f(x_k)\ ,\ \ k=0,1,\dots$$

Numerical realisation

$$u_k := x_{k+1}-x_k$$

$$A_{m\times n}(x_k):=f'(x_k)\equiv\begin{pmatrix} f_{11}\cdots\cdots\cdots f_{1n} \\ \cdot \qquad\qquad \cdot \\ \cdot \qquad\qquad \vdots \\ \cdot \qquad\qquad \cdot \\ f_{m1}\cdots\cdots\cdots f_{mn} \end{pmatrix}(x_k)\ \in M_{m\times n}^K$$

where $f_{ij}(x_k):=\dfrac{\delta f_i}{\delta x_j}(x_1^k,\dots,x_n^k),\ 1\le i\le m,\ 1\le j\le n$

partial derivations of f to x in x_k

$\Rightarrow$ (a) Solve the linear equation $\quad A_{m\times n}(x_k)\cdot u_k=-f(x_k)$

(b) Interation $\quad x_{k+1}=x_k+u_k$

System of Linear Algebraic Equations

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

in matrix notation: $\mathbf{A}\,\mathbf{x} = \mathbf{b}$

where A is the $m \times n$ coefficient matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & \ldots & a_{2n} \\ . & . & & & . \\ . & . & & & . \\ a_{m1} & a_{m2} & \ldots & \ldots & a_{mn} \end{pmatrix}, \qquad A_{m\times n} = (a_{ij}) \in M^K_{m\times n}$$

$x = (x_1,\ldots,x_n)^T \in K^n$ is the n-dimensional unknown column vector and

$b = (b_1,\ldots,b_m)^T \in K^m$ is the given m-dimensional column vector

$b = 0$: homogeneous system
$b \neq 0$: inhomogeneous system

Solution set $\quad S(A,b) := \{x \in K^n \mid A \cdot x = b\}$

$S(A,b) = \emptyset \qquad$ unsolvable

Two linear systems $A_1 \cdot x = b_1$, $A_2 \cdot x = b_2$ are equivalent if they have the same solution set:

$$S(A_1,b_1) = S(A_2,b_2).$$

How to solve a linear system ?

How to solve a linear system ?

I. Quadratic case $Ax = b$ with $A \in M^{K}_{n \times n}$ and $x, b \in K^n$

(A) $det\, A \neq 0$

Because of $det\, A \neq 0$ we have A^{-1} and the multiplication of both sides by A^{-1} gives

$$A^{-1}(Ax) = A^{-1}b$$
$$(A^{-1}A)x = A^{-1}b \quad \text{or} \quad Ix = A^{-1}b$$

and finally $x = A^{-1}b$ is the unique solution of the system.

$\Rightarrow$ (1) $det\, A \neq 0$?
(2) calculate the inverse A^{-1}
(3) $x = A^{-1}b$ is the unique solution

(B) $det\, A = 0$

Because of $det\, A = 0$ the inverse A^{-1} not exists.

some rows are linear dependent and so we have less independent rows than unknowns: either there are no solutions or infinitely many

$\Rightarrow$ case II

Numerical Method: Cramer´s Rule

We have seen that if A is invertible then the solution is $x = A^{-1}b$.

The Cramer´ s Rule avoid the use of the inverse of A.

(1) $D := det\ A \neq 0$?

(2) Let D_i the determinant of the matrix that result from the replacement of the *i*th column in A by the vector b.
Compute $D_1, \ldots, D_n$

(3) Solution vector is given by

$$x_1 = \frac{D_1}{D}\ ,\ x_2 = \frac{D_2}{D}\ ,\ \ldots\ ,\ x_n = \frac{D_n}{D}$$

notice: The Cramer´s rule is only of theoretical importance as an explicit expression of the solution.
It is not used for systems containing many equations.

II. Non-quadratic case $Ax = b$ with $A \in M^K_{m\times n}$ and $x \in K^n$, $b \in K^m$

(i) We consider the **homogeneous system** $Ax = 0$:

obviously: Every homogeneous system has a soltution $x = 0$.

$x \neq 0$ is a non-trivial solution if $det\, A = 0$.

If there are two non-trivial solutions x_1, x_2 than we have infinitely many solutions: $A\cdot(\alpha x_1 + \beta x_2) = \alpha(A\cdot x_1) + \beta(A\cdot x_2) = 0$.

(ii) We consider the **inhomogeneous system** $Ax = b$, $b \neq 0$:

The matrix $B := (Ab) \in M^K_{m\times(n+1)}$ is called the augmented matrix of the linear system.

Existence of Solutions $rk(A) = rk(B)$!

Uniqueness of solution $rk(A) = n$
infinitely many solutions $rk(A) = r < min(m,n)$

Transformation By finitely many elementary operations every inhomogeneous system can be transformed in an equivalent system of the form

$$
\begin{array}{llllll}
a_{11}x_1 + & \ldots & & & + a_{1n}x_n & = b_1 \\
0 & a'_{22}x_2 + & \ldots & & + a'_{2n}x_n & = b'_2 \\
 & \cdot & & & \cdot & \cdot \\
 & & & & \cdot & \cdot \\
0\ \ldots & 0 & a''_{qq}x_q + & \ldots & + a''_{qn}x_n & = b''_q \\
 & & & 0 & & = b''_{q+1} \\
 & & & \cdot & & \cdot \\
 & & & 0 & & = b''_n
\end{array}
$$

$\Rightarrow$ the basic of the use of some numerical methods like Gauss-Jordan Elimination

Numerical Method: Gauss-Jordan Elimination

in general case we have a large number of equations

so we need a solution procedure that amount to a systematic application of arithmetic steps that reduce our system to simpler and simpler form until the solution is evident

$\Rightarrow$ a **method of successive elimination:**

we transforme our system through a sequence of elementary operations into an upper triangular form (it is another system !)

To do so we have altered the solution set ?

Answer: No, since we have only used a finite number of elementary operations.

Gauss-Jordan Elimination steps via example

Consider the linear system

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 4 \\ x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 3x_2 - x_3 &= c \end{aligned} \quad , \quad c \in R$$

step 1 : eliminate the x_1 variable from the second through mth equation by adding (assume $a_{11} \neq 0$)

$-a_{21} / a_{11}$ times of the first equation to the second

....

$-a_{m1} / a_{11}$ times of the first equation to the ***m***th

$\Rightarrow$ an (equivalent) indented system of the form

$$\begin{aligned} 2x_1 + x_2 - 2x_3 &= 4 \\ \frac{3}{2}x_2 + 2x_3 &= 2 \\ \frac{3}{2}x_2 + 2x_3 &= c - 6 \end{aligned}$$

step 2: eliminate x_2 from the third through *m*th equation by adding (assume $a'_{22} \neq 0$)

$-a'_{32} / a'_{22}$ times of the second equation to the third

....

$-a'_{m2} / a'_{22}$ times of the second equation to the ***m***th

$\Rightarrow$ an (equivalent) indented system of the form

$$
\begin{aligned}
2x_1 + x_2 - 2x_3 &= 4 \\
\frac{3}{2}x_2 + 2x_3 &= 2 \\
0 &= c - 8
\end{aligned}
$$

step n-1: Continuing in this manner we eliminate $x_3, x_4, \ldots, x_{n-1}$ and the result is a system of the form

$$
\begin{array}{llllll}
a_{11}x_1 + & \ldots & & & + a_{1n}x_n & = b_1 \\
0 & a'_{22}x_2 + & \ldots & & + \mathrm{a}'_{2\mathrm{n}}x_n & = b'_2 \\
 & \cdot & & & \cdot & \cdot \\
 & & & & \cdot & \cdot \\
0 \ \ldots \ 0 & & a''_{qq}x_q + & \ldots & + a''_{qn}x_n & = b''_q \\
 & & & 0 & & = b''_{q+1} \\
 & & & \cdot & & \cdot \\
 & & & 0 & & = b''_n
\end{array}
$$

step n: apply back substitution starting in the qth equation

results:
i) if $q < n$ and $b''_{q+1}, \ldots, b''_n$ are not all zero the system has no solution
ii) if $q < n$ and $b''_{q+1}, \ldots, b''_n$ are all zero the system has a $(n - q)$-parameter family of solutions (infinity of solutions)
iii) if $q = n$ the system has a unique solution

back to our counter example:

There are two possibilities

1. if $c \neq 8$ then we have a contradiction and the solution set is empty; hence the considered system has no solution (because of equivalence): $S(A,b) = \varnothing$

2. if $c = 8$ then the third equation can be omitted and we set

   $x_3 = t$ as a free parameter (arbitrary)

   and then $x_2 = \dfrac{4(1-t)}{3}$, $x_1 = \dfrac{4+5t}{3}$

   $\Rightarrow$ $S(A,b) = \{ x \in R^3 | x = ((4+5t)/3, (4-4t)/3, t), t \in R \}$
   $\Rightarrow$ infinitely many solutions

2 The Eigenvalue Problem

2.1 Eigenvalues and Eigenvectors

What is an Eigenvalue Problem ? a concept of solving a linear equation; Important in many applications, for example in dynamic continous systems, control of processes

Solve the equation

$$Ax = \lambda x\,,\ x \neq 0$$

for a given square complex matrix $A \in M^{C}_{n\times n}$

Eigenvalue the value $\lambda \in C$ for which a solution $x \neq 0$ exist is called an eigenvalue of the matrix

Eigenvector the corresponding solution x is called an eigenvector of λ.

Let be the matrix of the form $A = B + iC$ with $B, C \in M^{R}_{n\times n}$ than we get

$$\begin{aligned} Ax = (B + iC)(u + iv) &= (Bu - Cv) + i(Cu + Bv) \\ &= \lambda(u + iv) = \lambda u + i\lambda v \end{aligned}$$

This is equivalent to the real system

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix}$$

and so we will consider only the eigenvalues of real matrices !

Remark: With the transformation $y = Ax$ the eigenvalue problem is eaquel to the question: Are there any vectors who are a multiple of themselves under the transformation.

How to find eigenvalues of a given matrix ?

The definition equation can be transformed in the homogeneous system

$$(A - \lambda I) \cdot x = 0 \;,\; x \neq 0$$

and so there are only non-trivial solutions x if $det(A - \lambda I) = 0$.

Characteristic Polynomial $p_n(\lambda) := det(A - \lambda I)$ is a polynomial in λ of degree n

(characteristic polynomial of the matrix A)

$\Rightarrow$ the eigenvalues of a real matrix $A \in M_{n \times n}^R$ are the solutions (roots) of the characteristic equation $det(A - \lambda I) = 0$

By the Fundamental Theorem of Algebra we get:

- a characteristic equation of degree n has exactly n roots and so the matrix $A \in M_{n \times n}^R$ has exactly n eigenvalues
- these eigenvalues may be real or complex and not necessarily distinct

Example We calculate the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix}$$

We get $p_3(\lambda) := det(A - \lambda I) = -\lambda^3 - 2\lambda^2 + 24\lambda = (-\lambda)(\lambda^2 + 2\lambda - 24)$

and the roots are $\lambda_1 = 0,\ \lambda_2 = 4,\ \lambda_3 = -6$

How to find eigenvectors of a given eigenvalue ?

Let be $\lambda \in C$ an eigenvalue of the matrix $A \in M_{n\times n}^{R}$ than all the non-trivial solutions $x \in C^n$ of the linear homogeneous system

$$(A - \lambda I) \cdot x = 0$$

are the corresponding eigenvectors of λ.

Example We determine the eigenvectors of the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 4$, $\lambda_3 = -6$ of the above matrix :

(a) $\lambda_1 = 0$: $(A - \lambda_1 I) \cdot x = A \cdot x = 0$

With elementary operations we receive the equivalent system

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We choose $x_3 = t$ and than we have $x_2 = -t$ and $x_3 = -t$.
Hence an eigenvector of $\lambda_1 = 0$ is $e^1 = (1,1,-1)^T$.

(b) $\lambda_2 = 4$: $(A - \lambda_2 I) \cdot x = \begin{pmatrix} -3 & 2 & 3 \\ 2 & -8 & -2 \\ 3 & -2 & -3 \end{pmatrix} \cdot x = 0$

With elementary operations we receive the equivalent system

$$\begin{pmatrix} -3 & 2 & 3 \\ 0 & -20 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and an eigenvector $e^2 = (1,0,1)^T$ of $\lambda_2 = 4$.

(c) $\lambda_3 = -6$: We get an eigenvector $e^3 = (-1,2,1)^T$ of $\lambda_3 = -6$.

Some important notions

Algebraic Multiplicity of an eigenvalue

If the characteristic polynomial has r distinct roots than it may be factorised in the form

$$p_n(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \dots (\lambda - \lambda_r)^{k_r} \text{ with } \sum_{i=1}^{r} k_i = n$$

and the root λ_i has the order k_i.

The integer k_i is called the algebraic multiplicity of λ_i

Geometric Multiplicity of an eigenvalue

The number of the corresponding linearly independent eigenvectors is called the geometric multiplicity of the eigenvalue.

Eigenspace of an eigenvalue

The space spanned by the linearly independent eigenvectors of an eigenvalue is called the eigenspace of an eigenvalue.

Some properties

The eigenvalues of a real symmetric matrix are real.

For a symmetric matrix $A \in M_{n \times n}^{R}$ there are exists n linearly Independent eigenvectors that are mutueally orthogonal.
That is they may be used as a basis of R^n.

Similarity matrices have the same eigenvalues: Let be λ an eigenvalue Of A and x a corresponding eigenvector. Furthermore let be B similar to A. Than $B = C^{-1}AC$ and with $y = C^{-1}x$ we have

$$By = C^{-1}ACy = C^{-1}ACC^{-1}x = C^{-1}Ax = C^{-1}\lambda x = \lambda C^{-1}x = \lambda y.$$

2.2 Numerical Methods

I. In applications it is often required to find the dominant eigenvalue and the corresponding eigenvector.

Power Method

Consider a matrix $A \in M^R_{n\times n}$ with n distinct eigenvalues $\lambda_1,...,\lambda_n$ and n corresponding linearly independent eigenvectors $e_1,...,e_n$.

We assume that the eigenvalues are ordered in the form $|\lambda_1|>|\lambda_2|>...|\lambda_n|$.
λ_1 is called the dominant eigenvalue of A.

Then any vector $x=(x_1,...,x_n)^T$ may be represented with the basis $\{e_1,...,e_n\}$ in the form $x=\alpha_1 e_1+...+\alpha_n e_n$ and we get

$$Ax = A(\alpha_1 e_1 + ... + \alpha_n e_n) = \alpha_1 Ae_1 + ... + \alpha_n Ae_n = \alpha_1\lambda_1 e_1 + ... + \alpha_n\lambda_n e_n .$$

Furthermore we have for any positive integer $k \in Z$

$$A^k x = \alpha_1\lambda_1^k e_1 + ... + \alpha_n\lambda_n^k e_n$$

and hence $$A^k x = \lambda_1^k\left[\alpha_1 e_1 + \alpha_2\left(\frac{\lambda_2}{\lambda_1}\right)^k e_2 + ... + \alpha_n\left(\frac{\lambda_n}{\lambda_1}\right)^k e_n\right] \quad (\text{ if } \lambda_1 \neq 0) .$$

Obviously we receive $\underline{\lim}_{k\to\infty} A^k x = \lambda_1^k\alpha_1 e_1$.

This rise in the iterative process

$$x_{k+1} = Ax_k = A(Ax_{k-1}) = A^2 x_{k-1} = ... = A^{k+1}x_0 \ , \ k = 0,1,... \ ,$$

where x_0 must be some arbitrary vector not orthogonal to e_1,

to compute the dominant eigenvalue λ_1 and the corresponding eigenvector e_1.

Disadvantage of the iteration: if $|\lambda_1|$ is large then $A^k x_0$ will become very large

$\Rightarrow$ scaling the vector x^k after each iteration, for example using the largest element $max(x^k)$ of x^k

resulting iterative process

$$y_{k+1} = Ax_k \ ,$$
$$x_{k+1} = \frac{y_{k+1}}{max(y_{k+1})} \ , \ k = 0,1,\ldots \ \text{with} \ x_0 := (1,1,\ldots,1)^T$$

Example We calculate the dominant eigenvalue λ_1 and the corresponding Eigenvector e_1 of a matrix with the power method in the above form.

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$y_1 = Ax_0 = (0,2,0)^T = 2(0,1,0)^T$, $max(y_1) = 2$ and $\lambda_1^{(1)} = 2$

$x_1 = \frac{1}{2} y_1 = (0,1,0)^T$

$y_2 = Ax_1 = (1,2,1)^T = 2(0.5,1,0.5)^T$, $max(y_2) = 2$ and $\lambda_1^{(2)} = 2$

$x_2 = \frac{1}{2} y_2 = (1/2,1,1/2)^T$

and so on we get

$y_3 = 2(0.25,1,0.25)^T$, $y_4 = 2(0.375,1,0.375)^T$, …,
$y_7 = 2(0.328,1,0.328)^T$, $y_8 = 2(0.336,1,0.336)^T$

$\Rightarrow$ approaching to $2(1/3,1,1/3)^T$ and so we have $\lambda_1 = 2$, $e_1 = \left(\frac{1}{3},1,\frac{1}{3}\right)^T$.

II. If the eigenvectors of a square matrix are known the transformation of the matrix to diagonal form can be carry out in some cases in a very effectiv manner.

Diagonalization method

Let be $A \in M^{R}_{n \times n}$ a matrix with n eigenvalues $\lambda_1,...,\lambda_n$ and a full set of n linearly independent eigenvectors $e_1,...,e_n$.

Modal matrix $M := (e_1,...,e_n)$ (the eigenvectors as columns)

Spectral matrix $\Lambda := diag(\lambda_1,...,\lambda_n)$

Then we have $A \cdot M = (Ae_1,...,Ae_n) = (\lambda_1 e_1,...,\lambda_n e_n) = M \cdot \Lambda$.

Since $det\, M \neq 0$ the matrix M is regular and the inverse M^{-1} exists. Hence we get the equation

$$M^{-1} \cdot A \cdot M = \Lambda .$$

⇒ Knowing the full set of linearly independent eigenvectors the matrix may be transformed in diagonal form with the eigenvalues (in the corresponding order)

⇒ On the other hand a matrix is uniquely determined once the eigenvalues and the full set of corresponding eigenvectors are known.

2.3 Functions of a Matrix

Let be $A \in M_{n\times n}^{R}$. Then we may built the matrix products $A^2, A^3, \ldots$ and $A^i \in M_{n\times n}^{R}$ for all $i = 0,1,2,\ldots$ with $A^0 := I$.

Power Series Function

$$f_m(A) = c_0 I + c_1 A + c_2 A^2 + \ldots + c_m A^m = \sum_{i=0}^{m} c_i A^i \quad \text{(polynomial function)}$$

$f_m : M_{n\times n}^{R} \to M_{n\times n}^{R}$ with $c_i \in R$ for every fixed $m \in N$

Consider $f_m(A) := \sum_{i=0}^{m} c_i A^i$ for $m = 1,2,\ldots$.

If the sequence $\{ f_m(A) \}$ converges for $m \to \infty$ to a constant matrix $G \in M_{n\times n}^{R}$ then we write in analog to the scalar power series

$$f(A) = \sum_{i=0}^{\infty} c_i A^i$$

and we call it the **power series function of the matrix** A.

Example Exponential matrix function

$$f(A) = e^{At} := I + \frac{A}{1!}t + \frac{A^2}{2!}t^2 + \ldots = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

Problem How to compute such functions of a square matrix ?

From where we will get the positive integer powers of the matrix ?

Let be $\lambda_1,...,\lambda_n$ the eigenvalues of a square matrix $A \in M^R_{n\times n}$ and $p_n(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + ... + c_1\lambda + c_0$ the characteristic polynomial of A.

Cayley-Hamilton Theorem

Then square matrix $A \in M^R_{n\times n}$ satisfies its own characteristic equation:

$$A^n + c_{n-1}A^{n-1} + ... + c_1 A + c_0 I = 0 \ .$$

$\Rightarrow$ positive integer powers of the square matrix $A \in M^R_{n\times n}$ may be expressed in terms of powers of A up to $n-1$:

$$A^k = \sum_{i=0}^{n-1} c_i A^i \ \text{ for all } k \geq n \ .$$

Let us consider at first the particular case of $n = 2$:

$A \in M^R_{2\times 2}$ with $p_2(\lambda) = \lambda^2 + c_1\lambda + c_0$

$\Rightarrow \quad A^2 + c_1 A + c_0 I = 0$ (Cayley-Hamilton)

$\Rightarrow \quad A^2 = -c_1 A - c_0 I$

$\Rightarrow \quad A^3 = -c_1 A^2 - c_0 A$ (by multiplying the above equation with A)

$\quad\quad = -c_1(-c_1 A - c_0 I) - c_0 A = (c_1^2 - c_0)A + c_1 c_0 I$

Obviously, this process we can continue, so that we will have the expression

$$A^k = b_0 I + b_1 A \ \text{ for any } k \geq 2 \ .$$

How to compute the constants $b_0, b_1 \in R$?

Since λ_1, λ_2 are eigenvalues of A we have the equations

$$p_2(\lambda_i) = \lambda_i^2 + c_1\lambda_i + c_0 = 0 \;,\; i = 1,2$$

and therefore
$$\lambda_i^2 = -c_1\lambda_i - c_0 ,$$
$$\lambda_i^3 = -c_1\lambda_i^2 - c_0\lambda_i = (c_1^2 - c_0)\lambda_i + c_1c_0$$

Proceeding in this way we get the analog expression for each eigenvalue

$$\lambda_i^k = b_0 + b_1\lambda_i \text{ for } i = 1,2 \text{ and any } k \geq 2$$

and from there we may calculate the constants $b_0, b_1 \in R$.

Remark: In the case the matrix A would have only one eigenvalue λ with the multiplicity $m = 2$, we can differentiate the equation $\lambda^k = b_0 + b_1\lambda$ with respect to λ to get a second equation for computing the 2 unknown b_0, b_1.

Hence, in the general case we have

$$f(A) = \sum_{i=0}^{\infty} c_i A^i = \sum_{i=0}^{n-1} b_i A^i \;,$$

where we obtain the coefficients $b_0, b_1, \ldots, b_{n-1}$ by solving the n equations

$$f(\lambda_i) = \sum_{i=0}^{n-1} b_i \lambda^i \;,\; i = 1,2,\ldots,n \;.$$

Remark: If the eigenvalue λ_i has the multiplicity m_i then the derivatives

$$\frac{d^k}{d\lambda_i^k} f(\lambda_i) = \frac{d^k}{d\lambda_i^k} \sum_{s=0}^{n-1} b_s \lambda_i^s \;,\; k = 1,2,\ldots,m_i - 1$$

are also satisfied by λ_i.

If $A \in M_{n \times n}^{R}$ possesses n linearly independent eigenvectors $e_1,\dots,e_n$ then we have an easier method of computing functions

$$f(A) = \sum_{i=0}^{k} c_i A^i \text{ of } A :$$

With the Modal matrix $M := (e_1,\dots,e_n)$ and the Spectral matrix $\Lambda := diag(\lambda_1,\dots,\lambda_n)$ we have the expression $M^{-1} \cdot A \cdot M = \Lambda$ and therefore

$$M^{-1} \cdot f(A) \cdot M = \sum_{i=0}^{k} c_i (M^{-1} A^i M) = \sum_{i=0}^{k} c_i (M^{-1} A M)^i = \sum_{i=0}^{k} c_i \Lambda^i$$

$$= \sum_{i=0}^{k} c_i diag(\lambda_1^i,\dots,\lambda_n^i)$$

$$= diag\left(\sum_{i=0}^{k} c_i \lambda_1^i, \sum_{i=0}^{k} c_i \lambda_2^i, \dots, \sum_{i=0}^{k} c_i \lambda_n^i \right)$$

$$= diag\big(f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)\big) .$$

Finally we get

$$f(A) = M \cdot diag\big(f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)\big) \cdot M^{-1} .$$

Example We compute $f(A) := A^k$ for the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$ with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$

and corresponding eigenvectors $e_1 = (1,-1)^T$, $e_2 = (1,-2)^T$.

Hence we get the modal matrix $M = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ and

the spectral matrix $\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$.

The inverse of M is $M^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$.

Since $f(-1) = (-1)^k$, $f(-2) = (-2)^k$ we get

$$f(A) = M \cdot \begin{pmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{pmatrix} \cdot M^{-1} = \begin{pmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ 2((-2)^k - (-1)^k) & 2(-2)^k - (-1)^k \end{pmatrix}$$

and therefore $$A^k = \begin{pmatrix} 2(-1)^k - (-2)^k & (-1)^k - (-2)^k \\ 2((-2)^k - (-1)^k) & 2(-2)^k - (-1)^k \end{pmatrix}.$$